

# The Dual Theorem concerning Aubert Line

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In this article we introduce the concept of Bobillier transversal of a triangle with respect to a point in its plan; we prove the Aubert Theorem about the collinearity of the orthocenters in the triangles determined by the sides and the diagonals of a complete quadrilateral, and we obtain the Dual Theorem of this Theorem.

## Theorem 1 (E. Bobillier)

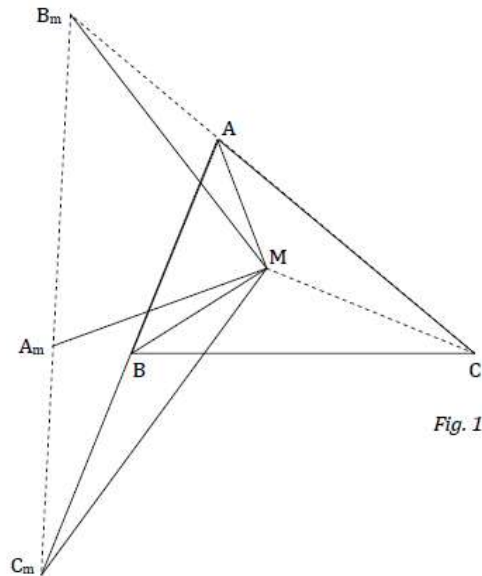
Let  $ABC$  be a triangle and  $M$  a point in the plane of the triangle so that the perpendiculars taken in  $M$ , and  $MA, MB, MC$  respectively, intersect the sides  $BC, CA$  and  $AB$  at  $Am, Bm$  and  $Cm$ . Then the points  $Am, Bm$  and  $Cm$  are collinear.

**Proof** We note that  $\frac{AmB}{AmC} = \frac{\text{aria}(BMAm)}{\text{aria}(CMAm)}$  (see Fig. 1).

$$\text{Area}(BMAm) = \frac{1}{2} \cdot BM \cdot MA_m \cdot \sin(\widehat{BMAm}).$$

$$\text{Area}(CMAm) = \frac{1}{2} \cdot CM \cdot MA_m \cdot \sin(\widehat{CMAm}).$$

Since



$$m(\widehat{CMAm}) = \frac{3\pi}{2} - m(\widehat{AMC}),$$

it explains that

$$\sin(\widehat{CMAm}) = -\cos(\widehat{AMC});$$

$$\sin(\widehat{BMAm}) = \sin\left(\widehat{AMB} - \frac{\pi}{2}\right) = -\cos(\widehat{AMB}).$$

Therefore:

$$\frac{AmB}{AmC} = \frac{MB \cdot \cos(\widehat{AMB})}{MC \cdot \cos(\widehat{AMC})} \quad (1).$$

In the same way, we find that:

$$\frac{BmC}{BmA} = \frac{MC}{MA} \cdot \frac{\cos(\widehat{BMC})}{\cos(\widehat{AMB})} \quad (2);$$

$$\frac{CmA}{CmB} = \frac{MA}{MB} \cdot \frac{\cos(\widehat{AMC})}{\cos(\widehat{BMC})} \quad (3).$$

The relations (1), (2), (3), and the reciprocal Theorem of Menelaus lead to the collinearity of points  $Am, Bm, Cm$ .

**Note** Bobillier's Theorem can be obtained – by converting the duality with respect to a circle – from the theorem relative to the concurrency of the heights of a triangle.

**Definition 1** It is called Bobillier transversal of a triangle  $ABC$  with respect to the point  $M$  the line containing the intersections of the perpendiculars taken in  $M$  on  $AM, BM$ , and  $CM$  respectively, with sides  $BC, CA$  and  $AB$ .

**Note** The Bobillier transversal is not defined for any point  $M$  in the plane of the triangle  $ABC$ , for example, where  $M$  is one of the vertices or the orthocenter  $H$  of the triangle.

**Definition 2** If  $ABCD$  is a convex quadrilateral and  $E, F$  are the intersections of the lines  $AB$  and  $CD$ ,  $BC$  and  $AD$  respectively, we say that the figure  $ABCDEF$  is a complete quadrilateral. The complete quadrilateral sides are  $AB, BC, CD, DA$ , and  $AC, BD$  and  $EF$  are diagonals.

### **Theorem 2 (Newton-Gauss)**

The diagonals' means of a complete quadrilateral are three collinear points. To prove Theorem 2, refer to [1].

**Note** It is called *Newton-Gauss Line* of a quadrilateral the line to which the diagonals' means of a complete quadrilateral belong.

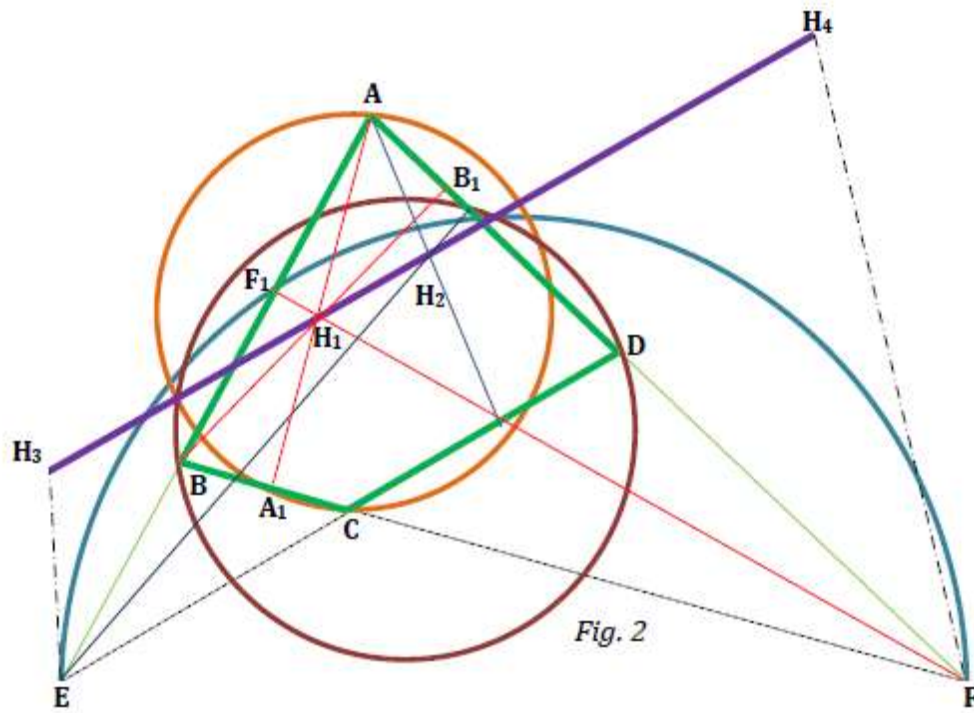
### **Teorema 3 (Aubert)**

If  $ABCDEF$  is a complete quadrilateral, then the orthocenters  $H_1, H_2, H_3, H_4$  of the triangles  $ABF, AED, BCE$ , and  $CDF$  respectively, are collinear points.

**Proof** Let  $A_1, B_1, F_1$  be the feet of the heights of the triangle  $ABF$  and  $H_1$  its orthocenter (see *Fig. 2*). Considering the power of the point  $H_1$  relative to the circle circumscribed to the triangle  $ABF$ , and given the triangle orthocenter's property according to which its symmetric to the triangle sides belong to the circumscribed circle, we find that:

$$H_1A \cdot H_1A_1 = H_1B \cdot H_1B_1 = H_1F \cdot H_1F_1.$$

This relationship shows that the orthocenter  $H_1$  has equal power with respect to the circles of diameters  $[AC]$ ,  $[BD]$ ,  $[EF]$ . As well, we establish that the orthocenters  $H_2, H_3, H_4$  have equal powers to these circles. Since the circles of diameters  $[AC]$ ,  $[BD]$ ,  $[EF]$  have collinear centers (belonging to the Newton-Gauss line of the  $ABCDEF$  quadrilateral), it follows that the points  $H_1, H_2, H_3, H_4$  belong to the radical axis of the circles, and they are, therefore, collinear points.



## Notes

1. It is called *the Aubert Line* or the line of the complete quadrilateral's orthocenters the line to which the orthocenters  $H_1, H_2, H_3, H_4$  belong.

2. The Aubert Line is perpendicular on the Newton-Gauss line of the quadrilateral (radical axis of two circles is perpendicular to their centers' line).

**Theorem 4 (The Dual Theorem of the Theorem 3)**

If  $ABCD$  is a convex quadrilateral and  $M$  is a point in its plane for which there are the Bobillier transversals of triangles  $ABC$ ,  $BCD$ ,  $CDA$  and  $DAB$ ; thereupon these transversals are concurrent.

**Proof** Let us transform the configuration in *Fig. 2*, by duality with respect to a circle of center  $M$ .

By the considered duality, the lines  $a, b, c, d, e$  and  $f$  correspond to the points  $A, B, C, D, E, F$  (their polars).

It is known that polars of collinear points are concurrent lines, therefore we have:  $a \cap b \cap e = \{A'\}$ ,  $b \cap c \cap f = \{B'\}$ ,  $c \cap d \cap e = \{C'\}$ ,  $d \cap f \cap a = \{D'\}$ ,  $a \cap c = \{E'\}$ ,  $b \cap d = \{F'\}$ .

Consequently, by applicable duality, the points  $A', B', C', D', E'$  and  $F'$  correspond to the straight lines  $AB, BC, CD, DA, AC, BD$ .

To the orthocenter  $H_1$  of the triangle  $AED$ , it corresponds, by duality, its polar, which we denote  $A'_1 - B'_1 - C'_1$ , and which is the Bobillier transversal of the triangle  $A'C'D'$  in relation to the point  $M$ . Indeed, the point  $C'$  corresponds to the line  $ED$  by duality; to the height from  $A$  of the triangle  $AED$ , also by duality, it correspond its pole, which is the point  $C'_1$  located on  $A'D'$  such that  $m(\widehat{C'MC'_1}) = 90^\circ$ .

To the height from  $E$  of the triangle  $AED$ , it corresponds the point  $B'_1 \in A'C'$  such that  $m(\widehat{D'MB'_1}) = 90^\circ$ .

Also, to the height from  $D$ , it corresponds  $A'_1 \in C'D'$  on  $C$  such that  $m(\widehat{A'MA'_1}) = 90^\circ$ . To the orthocenter  $H_2$  of the triangle  $ABF$ , it will correspond, by applicable duality, the Bobillier transversal  $A'_2 - B'_2 - C'_2$  in the triangle  $A'B'D'$  relative to the point  $M$ . To the orthocenter  $H_3$  of the triangle  $BCE$ , it will correspond the Bobillier transversal  $A'_3 - B'_3 - C'_3$  in the triangle  $A'B'C'$  relative to the point  $M$ , and to the orthocenter  $H_4$  of the triangle  $CDF$ , it will correspond the transversal  $A'_4 - B'_4 - C'_4$  in the triangle  $C'D'B'$  relative to the point  $M$ .

The Bobillier transversals  $A'_i - B'_i - C'_i$ ,  $i = \overline{1,4}$  correspond to the collinear points  $H_i$ ,  $i = \overline{1,4}$ .

These transversals are concurrent in the pole of the line of the orthocenters towards the considered duality.

It results that, given the quadrilateral  $A'B'C'D'$ , the Bobillier transversals of the triangles  $A'C'D'$ ,  $A'B'D'$ ,  $A'B'C'$  and  $C'D'B'$  relative to the point  $M$  are concurrent.

## References

[1] Florentin Smarandache, Ion Patrascu: „The Geometry of Homological Triangles”. The Education Publisher Inc., Columbus, Ohio, USA – 2012.

[2] Ion Patrascu, Florentin Smarandache: „Variance on Topics of Plane Geometry”. The Education Publisher Inc., Columbus, Ohio, USA – 2013.